SSE Riga - Maths Foundation

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February 19, 2022

Session 3 : Introduction to integration

Outline

- Indefinite integrals
- Area under a curve

Part I :

Indefinite integrals

Previous week : differentiation

The total cost function

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- Derivative : $C'(q) = 20q$
- This is the **marginal cost**
  "What is the increase of my total cost if I increase my production by one unit ?"

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Definition of the Indefinite integral

We write the indefinite integral

$$\int f(x)dx = F(x) + C,$$

where $F'(x) = f(x)$ (C is an arbitrary constant)

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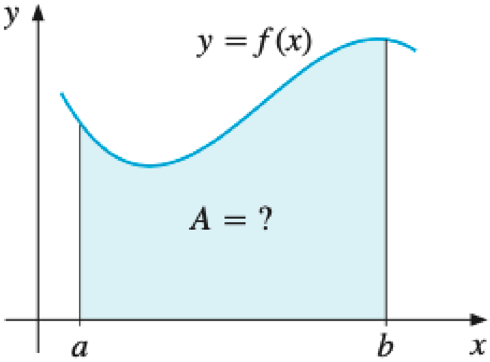
- $$\int af(x)dx = a\int f(x)dx \text{ (a is a constant)}$$
- $$\int \Big[f(x) + g(x)\Big]dx = \int f(x)dx + \int g(x)dx$$

Application

The GDP of a country is 80 billion euros and growing at the rate $4,5t^{-1/3}$ billion euros per year after t years. What is the GDP after 8 years ?

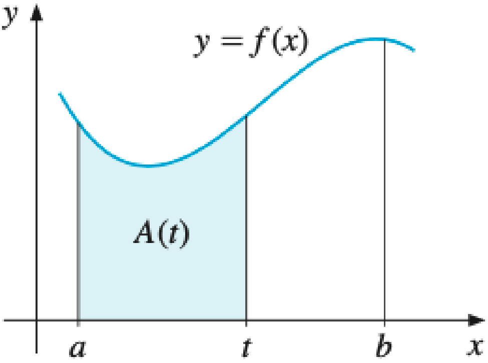
Part II :

Area under a curve - the definite integral



How to measure the area A?

- Let t be an arbitrary number in $[a; b]$



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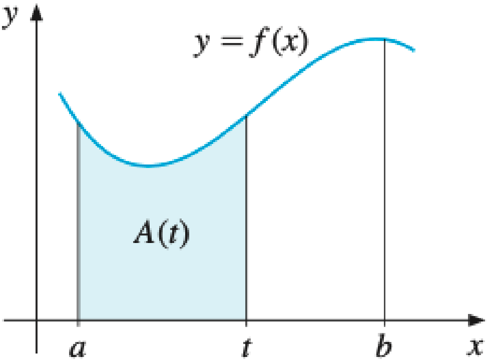

- $A(t)$ is the area under $y = f(x)$ over the interval $[a; t]$
  - $A(a) = 0$ and $A(b) = A$
  - When t increases, $A(t)$ increases

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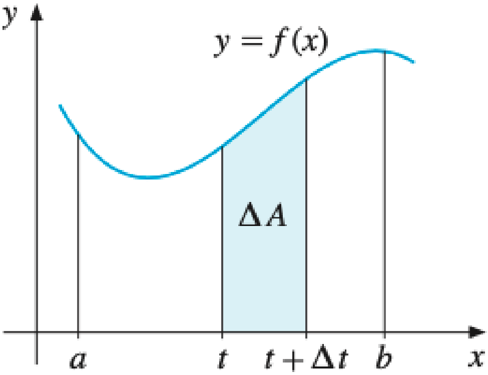

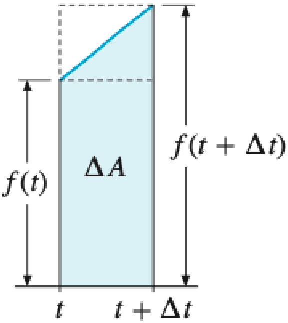

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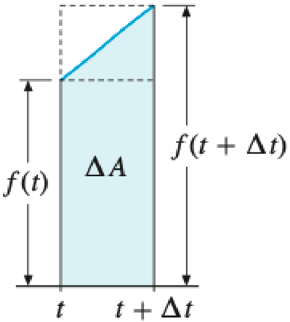

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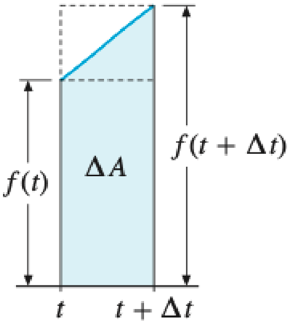

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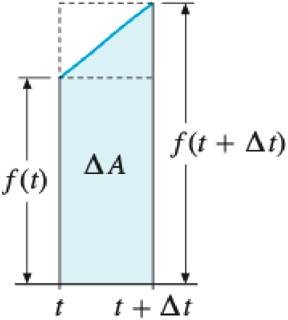

- Two observations :
    - ΔA cannot be larger than $f(t + \Delta t) \times \Delta t$
    - ΔA cannot be smaller than $f(t) \times \Delta t$
- As a consequence, we get :

$$f(t) \leq \frac{A(t + \Delta t) - A(t)}{\Delta t} \leq f(t + \Delta t)$$

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- Conclusion : the derivative of the area function $A(t)$ is $f(t)$, so the area function is the integral of $f(t)$!
- Ok, but which integral ?
  - It does not matter !

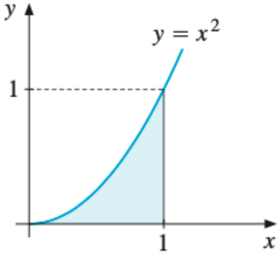
The definite integral

$$\int_a^b f(x)dx = \Big|_a^b F(x) = F(b) - F(a),$$

where F is any indefinite integral of f over an interval containing both a and b

Example

Area under the curve $y = x^2$ on the interval $[0; 1]$?



Example

$$\int_0^1 f(x)dx = \int_0^1 x^2 dx = \Big|_0^1 F(x) = \Big|_0^1 \frac{1}{3}x^3$$

$$= F(1) - F(0) = \frac{1}{3} \times 1 - 0 = \frac{1}{3}$$

Application

A company's marginal cost function is $MC(q) = \dfrac{75}{\sqrt{x}}$, where x is the number of units produced. Find the total cost of producing units 100 to 400.

Application

After t hours of work, a student can solve math problems at the rate of $r(t) = -t^2 + 4t + 5$ problems per hour. How many problems will this student process during the first three hours?

Thanks for your attention, and hope to see you soon!

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